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CONTROL SYSTEMS LABORATORY

A NOTE ON EQUATIONS

FOR A CLASS OF INTERACTION PROBLEMS

Report Number R-55

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Prepared by:

R. Duncan Luce

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Foreword

This paper deals with some aspects of the dynamics of information flow in networks. The experimental data referred to have been taken in a highly specialized laboratory situation. However, the resulting theory should apply to a wide range of conditions.

The MS was prepared during the Summer Study Session of 1953.

Henry Quastler

A NOTE ON EQUATIONS FOR A CLASS OF INTERACTION PROBLEMS

R. Duncan Luce

Bureau of Applied Social Research, Columbia University, N.Y,

1. Introduction

This note stems from an effort to write explicit equations for the probability distributions governing information flow, and hence the nodal information states in time, in a network whose nodes are human beings. As this type of problem, if not the particular details, is of some interest beyond experimental group psychology, it was deemed worthwhile presenting this formulation.

It has been concluded from experimental work that systems satisfying the following conditions are of interest:

1. They shall deal with more than one type of elementary particle, or, as we shall call it, element of information. The classical electric network equations describe systems in which only one type of particle -- the electron -- flows between the nodes.

2. Each time a node sends a message this serves to initiate a given time distribution which governs the time of sending of the subsequent message. This distribution will be taken to be independent of time and of the information content at the node.

3. Each message sent by a node shall include all the information present at the node at the time of sending, and the transmission of information will not result in any loss of information to the node.

4. The choice of destination of any message will be determined statistically according to given conditional probabilities, the conditions being the information which is present at the node at the time the message is sent.

The first, third and fourth statements seem acceptable without further discussion; however, the second is not obviously an appropriate assumption. Indeed, when this problem was first considered, the author felt the reasonable assumption was: the receipt of a message by a node initiated a given latency distribution which governed the next message sending of that node. It was shown [1] that this assumption led to an extremely complex problem. Since then an experimental group study has been completed [2] which strongly suggest that, at least for the case studied, assumption 2 is the more accurate of the two. For this experiment, the time data were examined from both points of view, and, though both gave reasonably clear structures, the receipt data were explained in terms of the sending data, and not conversely. The explanation was by no means comprehensive and rigorous, so we can only state that the experiment suggest assumption 2 is reasonable.

As we shall see, assumption 2 has the mathematical advantage that it leads to a problem we may formulate, which was not possible when the laternate assumption was made. The reader may compare the following sections with section IV. 9 of [1].

The severe abstraction of our model from most real systems is clear. In many systems the sources of messages in the recent past may alter the destination of a message in the near future; stimulation to send messages may depend on the content of (some) received messages; the latency distributions may be a function of information present at a node; etc.

2. The Problem

Let G and U be two given sets, G having n members labelled 1, 2, ..., n. We shall treat G as a communicating system of n nodes and U as the formal set of information with which G is dealing. The elements $\alpha \in U$ are essentially labels assigned to pieces of information which are indivisable in the context of the problem. For $i, j \in G$, $\alpha \in U$, $V \subset U$ let the following be given:

probability densities $f_i(t)$

real numbers τ_i

non-negative functions $g_i(\alpha, t)$ such that $\int_{-\infty}^{\infty} g_i(\alpha, t)dt \leq 1$

and sets of conditional probabilities $r_{ij}(V)$.

We assume that prior to τ_i no message is sent by node i, and that the first message sending is governed by $f_i(t-\tau_i)$. If i sends a message at time τ then the distribution of the next sending is governed by $f_i(t - \tau)$.

If node i sends a message at some time when it has the set of information V, then the probability that this message is sent to j is $r_{ij}(V)$. In addition, we assume that the message to j includes all the information i has, i.e., V.

Finally, we assume that node i receives the information α from the environment at a time determined by $g_i(\alpha, t)$. We make the added and reasonable assumption that there exists some t_o (which by a translation of the time scale may be taken to be 0) such that

$$g_i(\alpha,t) = 0 \text{ for all } i \in G, \alpha \in U, \text{ and } t < 0.$$

In this case we may replace the given system with the driving functions $g_i(\alpha,t)$ by an augmented system having only initial conditions. To each pair i and α such that $g_i(\alpha,t) \not\equiv 0$ introduce a new node, denoted $i(\alpha)$, such that

$$r_{i(\alpha)j}(V) = \begin{matrix} 1 \text{ if } i = j \text{ and } V = \{\alpha\} \\ 0 \text{ otherwise} \end{matrix}$$

$$r_{ji(\alpha)}(V) = 0 \text{ for all } j \text{ and } V$$

Let the node $i(\alpha)$ have the time characteristic $g_{i(\alpha)}(t) = g_i(\alpha,t)$ which is initiated at $t = 0$, but which, in contrast to the original nodes, does not have the property that a message sent causes the distribution to be reactivated.

In this augmented system the driving functions have vanished and we have the initial conditions that any node $i(\alpha)$ has the information set $\{\alpha\}$ at time 0 and the nodes $i \in G$ have no information at time 0.

Problem: given $f_i(t)$, τ_i, $r_{ij}(V)$, and $g_i(\alpha,t)$, where $i, j \in G, \alpha \in U, V \subset U$, determine the probability $P_i(\alpha,t)$ that node i has α at time t and the probability $P_i(V,t)$ that node i has exactly the subset V at time t.

3. The Time Characteristics

Mathematically, the most important feature of our assumptions is that the timing of message sendings from a given node is statistically independent of the behavior of the remainder of the system. This follows directly from the second major assumption. Without this conclusion, formulating the model is exceedingly difficult, but with it the formulation is comparatively simple.

Consider node i with latency $f_i(t)$ and initial activation τ_i which we may take to be 0 without loss of generality. The probability density that a message is sent at time t and that it is preceeded by n-1 sendings at times $t_1, t_2, \ldots t_{n-1}$ is

$$f_i(t_1)f_i(t_2-t_1)\ \ldots\ f_i(t-t_{n-1})$$

Thus, the density of the same occurrence but without specifying the times is

$$g_i^{(n)}(t) = \int_o^t \ldots \int_o^{t_3} \int_o^{t_2} f_i(t_1)f_i(t_2-t_1)\ldots f_i(t-t_{n-1})dt_1dt_2\ldots dt_{n-1} \qquad (1)$$

Finally, the density function $g_i(t)$ that node i sends a message at time t is given by summing equation 1 over all possible values of n,

$$g_i(t) = \sum_{n=1}^{\infty} g_i^{(n)}(t) = \sum_{n=1}^{\infty} \int_o^t \ldots \int_o^{t_3} \int_o^{t_2} f_i(t_1)f_i(t_2-t_1)\ldots f_i(t-t_{n-1})dt_1dt_2\ldots dt_{n-1} \qquad (2)$$

It is of interest that for the nodes $i \in G$, $g_i(t)$ plays exactly the same role as does $g_{i(\alpha)}(t) = g_i(\alpha,t)$ for the added nodes $i(\alpha)$. For the nodes $i(\alpha)$ we assume the original stimulation of the sending process occurs at $\tau_{i(\alpha)} = 0$, whereas for the other nodes no assumption as to the value of τ_i need be made. The most reasonable assumptions are either $\tau_i = 0$ or that it is the first value of t such that node i receives an input from one of the $i(\alpha)$. The latter assumption can lead to a complex problem. Be that as it may, in terms of the quantities $g_i(t)$ and $g_{i(\alpha)}(t)$ there is no distinction between the original and the added nodes, so in all further discussion we shall not distinguish between them in our notation. That is, G will be taken to be the augmented group.

Equation 2, while apparently quite formidable, is actually very simple in certain important cases. In general it has been shown [1] that if an organixm is stimulated at $t = 0$ and if the probability of a response in the interval $(t, t+\Delta t)$ is $\lambda(t)\Delta t$ as $\Delta t \to 0$, assuming no response has occured in the interval $(0,t)$, then the distribution of responses is

$$\lambda(t)e - \int_0^t \lambda(x)dx$$

if we let $\Lambda(t) = \int_0^t \lambda(x)dx$, then the family

$$\Lambda(t) = \lambda t - \ln \left(1 + \lambda t + \dots + \frac{(\lambda t)^k}{k}\right) ,$$

or equivalently the family of distributions

$$\frac{\lambda^{k+1} t}{k} \, C^{-\lambda t} ,$$

has been found to be empirically useful, particularly the cases $k = 0$ and 1 (1,2). For this family,

$$g_1^{(n)} t = \frac{\lambda^{n(k+1)}}{(k!)^n} e^{-\lambda t} \int_0^t \cdots \int_0^{t_3} \int_0^{t_2} t_1^{\,k} (t_2-t_1)^k (t_3-t_2)^k \cdots$$

$$(t-t_{n-1})^k dt_1 dt_2 \ldots dt_{n-1}$$

$$= \frac{\lambda^{n(k+1)}}{(k!)^n} e^{-\lambda t} \int_0^t \cdots \int_0^{t_3} t_2^{(2k-1)} B(k+1, k+1)\, (t_3-t_2)^k \cdots$$

$$(t-t_{n-1})^k\, dt_2 \ldots dt_{n-1}$$

$$= \frac{\lambda^{n(k+1)}}{(k!)^n} e^{-\lambda t} \int_0^t \cdots \int_0^{t_4} t_3^{(3k+2)} B(k+1, k+1)\, B(2(k+1, k+1)$$

$$(t_4-t_3)^k \ldots (t-t_{n-1})^k dt_3 \ldots dt_{n-1}$$

$$= \ldots$$

$$= \frac{\lambda^{n(k+1)}}{(k!)^n} e^{-\lambda t} \quad t^{\left[n(k+1)-1\right]} \prod_{i-1}^{n-1} B(i(k+1), k+1),$$

where $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ is the Beta function. Since k is an integer,

$$\prod_{i=1}^{n-1} B(i(k+1), k+1) = \frac{k!k!}{(2k+k)!} \cdot \frac{(2k+1)!k!}{(3k+2)!} \cdots \frac{(n-1)k+n-2\ !k!}{(nk+n-1)!} \,.$$

$$= \frac{(k!)^n}{(nk+n-1)!} \quad ,$$

so

$$g_1^{(n)}(t) = \frac{\lambda^{n(k+1)}\, t^{\left[n(k+1)-1\right]} e^{-\lambda t}}{(nk+n-1)!} \,.$$

Summing,

$$g_1(t) = \lambda e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{[n(k+1)-1]}}{[n(k+1)-1]!}$$

It only remains to evaluate the sum. It can be shown[3] that

$$\phi(x) = \sum_{n=0}^{\infty} \frac{x^{nj}}{(nj)!} = \frac{1}{j} \sum_{l=1}^{j} e^{\rho_j x},$$

where ρ_j are the j^{th} roots of unity. Taking the derivative of ϕ we have

$$\frac{d\phi(x)}{dx} = \sum_{n=1}^{\infty} \frac{x^{(nj-1)}}{(nj-1)!} = \frac{1}{j} \sum_{i=1}^{j} \rho_i e^{\rho_i x} \quad ;$$

whence,

$$g_i(t) = \frac{\lambda e^{-\lambda t}}{k+1} \sum_{i=1}^{k+1} \rho_i e^{\rho_i \lambda t},$$

where, now, the ρ_i are the $k+1^{st}$ roots of unity.

For k = 0, 1, 2 we may evaluate this in more familiar terms:

$k=0$: $g_1(t) = \lambda$

$k=1$: $g_1(t) = \frac{\lambda}{2}(1-e^{-2\lambda t})$

$k=3$: $g_1(t) = \frac{\lambda}{3}\left(1-e^{-\frac{3}{2}\lambda t}\left\{\cos\sqrt{\frac{3}{2}}\lambda t - \sqrt{3}\sin\sqrt{\frac{3}{2}}\lambda t\right\}\right)$

4. The Decision Characteristics

As we stated in the formulation of the assumptions, we take $P_1(\alpha,t)$ to be the probability that α will be among the pieces of information i has at time t, and $P_i(V,t)$ to be the probability that node i has exactly the set of information V at time t. First of all, it is clear that these two variables are related by

$$P_i(\alpha,t) = \sum_{\substack{\text{all } V \\ \text{such that} \\ \alpha \in V}} P_i(V,t); \tag{3}$$

hence, we need only develop equations for one of these variables. We shall deal with $P_i(\alpha,t)$.

The probability that node i has α at $t + \Delta t$ is equal, in the limit as $\Delta t \to 0$, to the probability that i had α at t added to the probability i did not have it at t but received it in the interval $(t, t+\Delta t)$, i.e.,

$$P_i(\alpha, t+\Delta t) \cong P_i(\alpha,t) + \left[1-P_i(\alpha,t)\right] \varphi_i(\alpha,t,\Delta t), \tag{4}$$

where $\varphi_i(\alpha,t,\Delta t)$ is the probability i receives α in $(t, t+\Delta t)$. Of course, $\varphi_i(\alpha,t,\Delta t)$ is equal to 1 minus the probability i did not receive α in the interval. The probability of no receipt of α is the product that each of the other members of G did not send α. In symbols

$$\varphi_i(\alpha,t,\Delta t) = 1 - \prod_{j \neq i} \left[1 - \sum_{\substack{\text{all } V \\ \text{such that} \\ \alpha \in V}} P_j(V,t)\, r_{ji}(V)\, g_i(t)\Delta t\right],$$

or expanding the product as a sum

$$\varphi_i(\alpha,t,\Delta t) = \sum_{j \neq i} \sum_{\substack{V \\ \text{such that} \\ \alpha \in V}} P_j(V,t)\, r_{ji}(V)\, g_i(t)\Delta t + \text{terms in } \Delta t^2 \text{ or higher.} \tag{5}$$

Substituting equation 5 into equation 4 we have

$$P_i(\alpha, t+\Delta t) \cong P_i(\alpha, t) + \left[1 - P_i(\alpha, t)\right] \left[\sum_{j \neq i} \sum_{\substack{V \text{ such that} \\ \alpha \in V}} P_j(V,t)\, r_{ji}(V)\, g_j(t)\Delta t + \text{terms in } \Delta t^2 \text{ of higher} \right]. \quad (6)$$

Rewriting equation 6, dividing by Δt, and taking the limit as $\Delta t \to o$, we obtain the final result

$$\frac{dP_i(\alpha, t)}{dt} = \lim_{\Delta t \to o} \left\{ \frac{P_i(\alpha, t+\Delta t) - P_i(\alpha, t)}{\Delta t} \right\}$$

$$= \left[1 - P_i(\alpha, t)\right] \left[\sum_{j \neq i} \sum_{\substack{V \text{ such} \\ \text{that} \\ \alpha \in V}} P_i(V,t)\, r_{ji}(V)\, g_j(t) \right]. \quad (7)$$

Thus, our problem is formulated as equations (2), (3) and (7).

It is quite apparent that a general solution to this system of equations in terms of known functions is out of the question. To the person interested in specific answers two courses are open: either numerical solutions to specific cases using the available computation machinery of analytic solutions or certain simple cases, which in certain circumstances may be pieced together to obtain solutions to more complicated problems.

In the next section we deal with one special case for which an analytic solution is possible.

5. Two-Person Symmetric Case

Consider a system of two nodes, 1 and 2, which may interact or send to an outside sink. Suppose the situation is completely symmetric in the following sense: At $t = 0$ node 1 receives the information α and node 2 the information β, and

$$P_1(\beta, t) = P_2(\alpha, t) = P(t)$$

$$f_1(t) = f_2(t) = f(t)$$

$$\tau_1 = \tau_2 = \tau$$

$$r_{12}(\alpha \cup \beta) = r_{21}(\alpha \cup \beta) = r_1$$

$$r_{12}(\alpha) = r_{21}(\beta) = r_o$$

and, of course,

$$P_1(\alpha, t) = P_2(\beta, t) = 1$$

$$P(0) = 0$$

$$g_1(t) = g_2(t) = g(t)$$

Under these conditions the system of equations (7) reduces to

$$\frac{dP}{dt} = \left[1 - P\right] \left[\sum_V P(V,t)\, r_{ij}\,(V) g(t)\right].$$

But since

$$\sum_V P(V,t) r_{1j}(V) = r_o + (r_1 - r_o) P(t),$$

we have

$$\frac{dP}{dt} = \Big[1 - P\Big] \quad \Big[r_o + (r_1 - r_o)\Big] \quad P \qquad g(t).$$

If we let $u = 1-P$, then

$$\int_{u(o)}^{u(t)} \frac{-du}{u\,[r_1+(r_o-r_1)u]} = \int_o^t g(x)dx = \frac{1}{r_1}\left[\left\{l_n \frac{r_1+(r_o-r_1)u}{u}\right\}\right]_{u(o)}^{u(t)}$$

Noting that $u(0) = 1-P(0) = 1$,

$$\ln \left\{\frac{r_1+(r_o-r_1)u}{r_o u}\right\} = r_1 \int_o^t g(x)dx,$$

or

$$P(t) = \frac{r_o\left\{\exp\left(r_1 \int_o^t g(x)dx\right) - 1\right\}}{r_1 - r_o + r_o\left(\exp\ r_1 \int_o^t g(x)dx\right)}, \quad t \geq o$$

$$= 0 \qquad , \quad t < o$$

is the solution.

To be a little more explicit, if we assume $\tau = 0$ and $f(t) = \lambda e^{-\lambda t}$, then from section 3 we know $g(s) = \lambda$, for $t \geq 0$, hence

$$= \frac{r_o(e^{r_1 \lambda t} - 1)}{r_1 - r_o + r_o e^{r_1 \lambda t}}, \qquad t \geq 0.$$

$$= 0 \qquad , \qquad t < 0.$$

To indicate how quickly troubles arise, suppose we consider the same two node case without the condition of symmetry. Then the system, e.g. (7), becomes

$$\frac{dP_1(\beta_\alpha, \ t)}{dt} = \left[1-P_1(\beta_\alpha, t)\right] \left[r_{21}(\beta) + \left\{r_{21}(\alpha \cup \beta) - r_{21}(\beta)\right\} P_2(\alpha, t)\right] g_2(t)$$

$$\frac{dp_2(\alpha, t)}{dt} = \left[1-P_2(\alpha, t)\right] \left[r_{12}(\alpha) + \left\{r_{12}(\alpha \cup \beta) - r_{12}(\alpha)\right\} P_1(\beta, t)\right] g_1(t),$$

which amounts to a non-linear second order equation with non-constant coefficients.

References

1. Christie, L. S., Luce, R. D., and Macy, J., Jr., "Communication and Learning in Task-Oriented Groups," Research Laboratory of Electronics Technical Report 231, MIT (1952).

2. Hay, Harvie, "The Effects of Certain Communication Changes on Task-Oriented Groups," Ph. D. Thesis, Department of Economics, MIT., 1953.

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2. The author wishes to express his appreciation to Lee S. Christie and J. Macy, Jr. of M.I.T. for their helpful discussions of this problem.

3. Albert Blank of the University of Illinois pointed out the closed form of this series to the author. In outline his proof depends on the fact that

$$\sum_{i=1}^{j} (\rho_i)^k = 0$$ when ρ_i are the $j^{\underline{th}}$ roots of unity, and k is any positive integer less than j. Expanding $e^{\rho_i x}$ in the usual exponential series, regrouping, and using the above fact, yields the result.